Econ. Theory 4, 677–688 (1994)

Economic Theory

Robust ergodic chaos in discounted dynamic optimization models

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Received: April 7, 1993; revised version September 22, 1993

Summary. We identify a family of discounted dynamic optimization problems in which the immediate return function depends on current consumption, capital input and a taste parameter. The usual monotonicity and concavity assumptions on the return functions and the aggregative production function are verified. It is shown that the optimal transition functions are represented by the "quadratic family", well-studied in the literature on chaotic dynamical systems. Hence, Jakobson's theorem can be used to throw light on the issues of robustness of ergodic chaos and sensitive dependence on initial conditions.

0. Introduction

It is by now well-known that a variety of models in economics gives rise to discrete time, *non-linear* processes of the form

$$x_{t+1} = h(x_t) \tag{0.1}$$

where the function h satisfies the Li–Yorke condition for "chaotic" or "complex" behavior. Besides the *relative abundance* of examples of chaos, yet another theme has rightly been stressed: *quite simple models* of economic theory may lead to such examples.

While formal results on chaotic processes have been derived primarily in the last ten years or so, interest in non-linear difference equations can be traced back at least to Samuelson's *Foundations* (pp. 302–310). Interestingly enough, Samuelson also speculated on a theory of *comparative dynamics*, which will "include the theory of comparative statics as a special case", but "will cover a much richer terrain." The central notion of *comparative dynamics* is that "we change something, and investigate the effect of this change on the whole motion or behavior of the economic system." The changes Samuelson talked about included (i) changes in initial conditions and (ii) changes in some parameter affecting the system. To this effect we can explicitly introduce a "parameter" μ and study a family of dynamic processes

$$x_{t+1} = h_\mu(x_t). \tag{0.2}$$

Now, admitting the possibility or even pervasiveness of complicated behavior, one can ask a natural question of interest to mathematical modelling of social phenomena: "to what extent is chaotic behavior robust?" If a process like (0.1) displays complexity due to some accidental matching of parameters, then clearly one need not take such behavior seriously. More generally, given a particular definition of complicated behavior, if the family (0.2) displays such behavior for a "negligible" set of parameter values, one can argue that the "typical" model is still "well-behaved".

In this paper we continue the investigation of the properties of optimal programs in discounted dynamic optimization models studied earlier (Majumdar and Mitra (1991)). Our focus is primarily on *ergodic chaos.* We attempt to throw light on the issue of its robustness by using a parametric variation approach in the spirit of (0.2). A family of discounted dynamic optimization problems is identified in which the one period return function depends on current consumption, capital input and some taste parameter μ. It is shown that this family of optimization problems gives rise to the dynamical system represented by the "quadratic family"; that is, for each value of the parameter μ, the optimal transition function is given by $h_\mu(x) = \mu x(1-x)$.

Using this result, it is shown that *the set of parameter values generating optimal programs that display ergodic chaos has positive Lebesgue measure.* The basic mathematical result that we apply is due to Jakobson (1981) [see also Benedicks and Carleson (1985) and Carleson (1991)].[1]

By using Jakobson's theorem one can also throw light on the issue of *sensitive dependence* of the processes (0.2) on initial conditions in the strong sense of Guckenheimer (1979). Roughly speaking, for a given value of μ, the process (0.2) possesses sensitive dependence on initial conditions if there is some positive number such that in *every* neighborhood of an initial x there exists a point such that its image under a finite number of iterations separates from that of x by that positive number. Here, again, the size of such a set of initial conditions can be an issue: if it is of Lebesgue measure zero, it is "negligible". The definition of Guckenheimer requires that such sensitivity holds on a set of initial conditions with positive Lebesgue measure. Jakobson's result also implies that *the set of parameter values generating optimal programs that display Guckenheimer's sensitive dependence has positive Lebesgue measure.*

This quadratic family of maps has been the focus of attention of "bifurcation analysis", which examines, roughly speaking, how the limit sets (asymptotic behavior) of a family of dynamic processes (like (0.2)) change as the parameter μ changes. Given this literature (emphasizing especially the "period-doubling route to chaos") our family of dynamic optimization problems allows us to also

[1] The question of whether the family of dynamic optimization problems displaying robust ergodic chaos is itself non-negligible in an appropriate class of dynamic optimization problems is an important one. We do not pursue this question in this paper. It is worth noting, though, that Jakobson's result actually holds for some families of maps "close to" the quadratic family in the C^3 metric. (see Jakobson (1981, p. 40)). This suggests that the answer to the above question might be in the affirmative in a "sufficiently smooth" class of dynamic optimization problems.

explore – in the spirit of Samuelsonian comparative dynamics – how the long-run behavior of optimal programs undergoes qualitative changes as the taste parameter μ varies.

In Section I we collect some definitions. The family of dynamic optimization problems is formally set up in Section II. The main result on robustness of ergodic chaos and sensitive dependence on initial conditions is presented in Section III. The steps needed to show that the dynamical system generated by this family of dynamic optimization problems is given by the quadratic family involve some detailed calculations and verifications (more refined than those in Majumdar and Mitra (1991)). These are relegated to an appendix.

It should perhaps be stressed that while we use an explicit parametric variation in the spirit of (0.2) in this paper, our earlier result on robust topological chaos rested on varying the one period return and production functions on an open set in the function spaces (and the discount factor on an open interval). We note that while the question of robustness of topological chaos has been discussed in a few other environments, we do not know of any attempt to characterize robust ergodic chaos or sensitive dependence. A complete list of related reference is already given in Majumdar and Mitra (1991), which contains a detailed economic interpretation of the basic model.

I. Technical preliminaries

Consider a probability space (X, Σ, ν) and a Σ-measurable map $h: X \to X$; ν is *invariant under h* if $\nu(E) = \nu(h^{-1}(E))$ for all $E \in \Sigma$; ν is *ergodic* if "$E \in \Sigma$, $h^{-1}(E) = E$" implies "$\nu(E) = 0$ or $\nu(E) = 1$". Let X be a closed interval $[\alpha, \beta]$ of the real line with $\alpha < \beta$. We say that $h: X \to X$ exhibits *ergodic chaos* if there is an ergodic invariant measure that is absolutely continuous with respect to the Lebesgue measure. It is known, for example, that

$$h(x) \equiv 4x(1-x)$$

exhibits ergodic chaos.

If $x \in X$, the sequence $\tau(x) \equiv (h^j(x))_{j=0}^{\infty}$ is called the *trajectory* from (the initial condition) x. The *orbit* from x is the set $\gamma(x) \equiv \{y: y = h^j(x) \text{ for some } j \geq 0\}$. The asymptotic behavior of a trajectory from x is described by the *limit set*, which is defined as the set of all limit points of $\gamma(x)$, and is denoted by $\omega(x)$.

A point $x \in X$ is a *fixed point* of h if $h(x) = x$. A point $x \in X$ is called *periodic* if there is $k \geq 1$ such that $h^k(x) = x$. The smallest such k is the *period* of x. Note that if $x \in X$ is a periodic point, then $\omega(h^j(x)) = \gamma(x)$ for every $j = 0, 1, \ldots$. A periodic point $\bar{x} \in X$ is *stable* if there is an open interval V (in X) containing $\bar{x}$, such that $\omega(x) = \gamma(\bar{x})$ for all $x \in V$. [In this case we also say that the periodic orbit $\gamma(\bar{x})$ is stable.]

Consider the iterations

$$h^0(x) = x \quad \text{and} \quad h^k(x) = h[h^{k-1}(x)] \quad \text{for } k \geq 1. \tag{1.1}$$

The process

$$x_{t+1} = h(x_t) \tag{1.2}$$

has *sensitive dependence on initial conditions* (Guckenheimer (1979)) if there are a set $Y \subset X$ of positive Lebesgue measure and an $\varepsilon > 0$ such that given any $x \in Y$ and any neighborhood U of x, there is $y \in U$ and $n \geq 1$ such that $|h^n(x) - h^n(y)| > \varepsilon$.

To address the question of robustness, let (h_μ) be the "quadratic" family of maps, where $\mu \in A \equiv [1, 4]$ and $h_\mu: [0, 1] \to [0, 1]$ is defined as:

$$h_\mu(x) \equiv \mu x(1 - x). \tag{1.3}$$

Let $\Lambda \equiv \{\mu \in A: h_\mu$ exhibits ergodic chaos$\}$. The basic result of Jakobson characterizes this set Λ: it is uncountable *and* non-negligible. More precisely, let us state:

Jakobson's Theorem (Theorem B and remarks on p. 40 of Jakobson [1981]):
The set Λ, such that for each $\mu \in \Lambda$ the map h_μ exhibits ergodic chaos, has positive Lebesgue measure; also, h_μ has sensitive dependence in the sense of Guckenheimer.

The following result from bifurcation analysis summarizes the period-doubling route to chaos for the family of maps (1.3).

Bifurcation Theorem
There is a sequence $(\mu_k)_{k=0}^{\infty}$ of bifurcation values of μ, such that
(i) $\mu_0 = 1$, $\mu_k < \mu_{k+1}$ *for* $k = 0, 1, 2, 3, \ldots$
(ii) $\mu_k \to \mu_\infty$ *as* $k \to \infty$, *where* $\mu_\infty \simeq 3.569946$.
(iii) *For $\mu \in (\mu_k, \mu_{k+1})$, there is a stable periodic point $x(\mu)$, with period 2^k, such that for almost every $x \in X$, $\omega(x) = \gamma(x(\mu))$.*

II. A family of dynamic optimization problems

We consider a *family* of intertemporal economies, indexed by a parameter μ, where $\mu \in A = [1, 4]$. Each economy in this family has the same *production function*, $f: \Re_+ \to \Re_+$ and the same *discount factor* $\delta \in (0, 1)$. The economies in this family differ in the specification of their *welfare functions*, $w: \Re_+^2 \times A \to \Re_+$ [depending on the parameter value of $\mu \in A$ that is picked].

The following assumptions on f are used:

(F.1) $f(0) = 0$.
(F.2) *f is non-decreasing, continuous and concave on $\Re_+$.*
(F.3) *There is $\mathbf{K} > 0$, such that $f(x) < x$ for all $x > \mathbf{K}$, and $f(x) > x$ for all $0 < x < \mathbf{K}$.*

A *program* from $x \geq 0$ is a sequence (x_t) satisfying

$$x_0 = x, \quad 0 \leq x_t \leq f(x_{t-1}) \quad \text{for } t \geq 1$$

The *consumption* sequence (c_t), generated by a program (x_t) is given by

$$c_t = f(x_{t-1}) - x_t\ (\geq 0) \quad \text{for } t \geq 1$$

It is standard to verify that for any program (x_t) from $x \geq 0$, we have x_t, $c_{t+1} \leq \mathbf{K}(x) = \max(\mathbf{K}, x)$ for $t \geq 0$.

Given any $\mu \in A$, the following assumptions on $w(\cdot, \mu)$ are used:

(W.1) *$w(c, x, \mu)$ is non-decreasing in c given x, and non-decreasing in x, given c.*
(W.2) *$w(c, x, \mu)$ is continuous on $\Re_+^2$.*
(W.3) *$w(c, x, \mu)$ is concave on $\Re_+^2$.*

In defining "optimality" of a program, we note that the notion has to be economy specific. Since we can keep track of the economies by simply noting its μ value, we find it convenient to refer to the appropriate notion of optimality as μ-optimality.

Given any $\mu \in A$, a program $(\hat{x}_t)$ from $x \geq 0$ is *μ-optimal* if

$$\sum_{t=0}^{\infty} \delta^t w(\hat{c}_{t+1}, \hat{x}_t, \mu) \geq \sum_{t=0}^{\infty} \delta^t w(c_{t+1}, x_t, \mu)$$

for every program (x_t) from x.

Define a set $Y \subset \mathfrak{R}^2_+$ by

$$Y = \{(c, x) \in \mathfrak{R}^2_+ : c \leq f(x)\}$$

For much of our discussion of μ-optimal programs, what is crucial is the behavior of $w(\cdot, \mu)$ on Y (rather than on $\mathfrak{R}^2_+$). We now proceed to assume:

(W.4) Given any $\mu \in A$, $w(c, x, \mu)$ is strictly increasing and strictly concave in c given x, on the set Y.

Standard arguments ensure that given any $\mu \in A$, there is a μ-optimal program from every $x \geq 0$. Assumptions (F.2), (W.3), and (W.4) ensure that a μ-optimal program is unique.

Given any $\mu \in A$, we define a *value function*, $V: \mathfrak{R}_+ \times A \to \mathfrak{R}$ by

$$V(x, \mu) = \sum_{t=0}^{\infty} \delta^t w(\hat{c}_{t+1}, \hat{x}_t, \mu)$$

where $(\hat{x}_t)$ is the μ-optimal program from $x \geq 0$.

Since there is a unique μ-optimal program from every $x \geq 0$, one can define an *optimal transition function* $h: \mathfrak{R}_+ \times A \to \mathfrak{R}_+$ by

$$h_\mu(x) = \hat{x}_1$$

where $(\hat{x}_t)$ is the μ-optimal program from $x \geq 0$. It is easily checked that this definition also implies that for all $t \geq 0$, we have

$$\hat{x}_{t+1} = h_\mu(\hat{x}_t)$$

III. A family displaying robust ergodic chaos and sensitive dependence

Consider now a family of economies, where f, δ and w are specified as follows:

$$f(x) = \begin{cases} (16/3)x - 8x^2 + (16/3)x^4 & \text{for } x \in [0, 0.5] \\ 1 & \text{for } x \geq 0.5 \end{cases} \tag{3.1}$$

$$\delta = 0.0025$$

The function w is specified in a more involved fashion. To case the writing, denote $L \equiv 98$, $a \equiv 425$. Also, denote by I the closed interval $[0, 1]$, and define the function $\theta: I \times A \to I$ by

$$\theta(x, \mu) = \mu x(1 - x) \quad \text{for } x \in I,\ \mu \in A$$

and $u: I^2 \times A \to \Re$ by

$$u(x, z, \mu) = ax - 0.5Lx^2 + z\theta(x, \mu) - 0.5z^2 - \delta[az - 0.5Lz^2 + 0.5\theta(z, \mu)^2] \quad (3.2)$$

Define a set $\boldsymbol{D} \subset I^2$ by

$$\boldsymbol{D} = \{(c, x) \in \Re_+ \times I : c \leq \boldsymbol{f}(x)\}$$

and a function $w: \boldsymbol{D} \times A \to \Re_+$ by

$$w(c, x, \mu) = u(x, \boldsymbol{f}(x) - c, \mu) \quad \text{for } (c, x) \in \boldsymbol{D} \text{ and } \mu \in A \quad (3.3)$$

We now extend the definition of $w(\cdot, \mu)$ to the domain Y. For $(c, x) \in Y$ with $x > 1$ [so that $\boldsymbol{f}(x) = 1$, and $c \leq 1$], define

$$w(c, x, \mu) = w(c, 1, \mu) \quad (3.4)$$

Finally, we extend the definition of $w(\cdot, \mu)$ to the domain $\Re^2_+$. For $(c, x) \in \Re^2_+$ with $c > \boldsymbol{f}(x)$, define

$$w(c, x, \mu) = w(\boldsymbol{f}(x), x, \mu) \quad (3.5)$$

It can be checked (see Appendix A) that for the above specifications, $\boldsymbol{f}$ satisfies (F.1)–(F.3), and given any $\mu \in A$, $w(\cdot, \mu)$ satisfies (W.1)–(W.4).

We observe that $w(c, x, \mu) \geq w(0, 0, \mu)$[by (W.1)] $= u(0, \boldsymbol{f}(0) - 0, \mu) = u(0, 0, \mu) = 0$, for all $(c, x) \in \Re^2_+$. Thus $w(\cdot, \mu)$ maps from $\Re^2_+$ to $\Re_+$. Also, for all $(c, x) \in \Re^2_+$, $w(c, x, \mu) \leq w(c, 1, \mu) \leq w(\boldsymbol{f}(1), 1, \mu) = w(1, 1, \mu)$.

We verify (in Appendix B) the following:

Theorem 3.1

The optimal transition functions for the family of economies $(\boldsymbol{f}, w(\cdot, \mu), \delta)$ are given by

$$h_\mu(x) = \mu x(1 - x) \quad \text{for all } x \in I \quad (3.6)$$

Hence, by Jakobson's Theorem, the family exhibits robust ergodic chaos and sensitive dependence on initial conditions.

Using (3.6) of Theorem 3.1, and applying the Bifurcation Theorem, we can conclude that as the taste parameter μ changes, for instance, from $1 < \mu < \mu_1$ to $\mu_1 < \mu < \mu_2$, the long-run behavior of the typical optimal program changes from convergence to a stable fixed point of $h_\mu(x)$ to convergence to a stable periodic point of period two. That is, the long-run dynamic behavior of optimal programs experiences a bifurcation (a distinct qualitative change) as the taste parameter μ crosses the value μ_1. The same is, of course, true of the other bifurcation values μ_k for $k > 1$ as each successive bifurcation value gives rise to a stable periodic point of a period which is double that of the previous bifurcation value.

Appendix A
Verification of the assumptions

The production function

Recall that the production function, $\boldsymbol{f}: \Re_+ \to \Re_+$ is given by

$$f(x)=\begin{cases}(16/3)x-8x^2+(16/3)x^4 & \text{for } x\in[0,0.5)\\ 1 & \text{for } x\geq 0.5\end{cases}$$

Note that $f(0)=0$, and $f(x)\to 1$ as $x\uparrow(1/2)$, so that f is clearly continuous on $\Re_+$. Next, note that for $0\leq x<0.5$, f is C^2, and

$$f'(x)=(16/3)-16x+(64/3)x^3$$
$$f''(x)=-16+64x^2$$

As $x\uparrow(1/2)$, $f'(x)\to 0$. Thus, we can conclude that f is C^1 on $\Re_+$. As $x\uparrow(1/2)$, $f''(x)\to 0$, so repeating the above argument, f is C^2 on $\Re_+$. It is clear that $f''(x)<0$ for $0\leq x<0.5$, and so $f'(x)>0$ for $0\leq x<0.5$ [since $f'(1/2)=0$]. Thus, f is non-decreasing and concave on $\Re_+$.

To verify (F.3), note that the concavity of f and $f(0)=0$ ensure that $[f(x)/x]$ is non-increasing for $x>0$. Since $f(1/2)>(1/2)$, we have $f(x)>x$ for $0<x\leq(1/2)$. And for $(1/2)<x<1$, $f(x)=1$, so $f(x)>x$ for $(1/2)<x<1$ as well. Thus, defining $\mathbf{K}=1$, we have $f(x)>x$ for $0<x<\mathbf{K}$, and $f(x)=1<x$ for all $x>\mathbf{K}$.

We now note an important property of f, viz,

$$f(x)\geq\theta(x,\mu)\quad\text{for all } x\in I,\ \mu\in A$$

Since $f(x)=1$ for $x\in[0.5,1]$, and $\theta(x,\mu)\in I$ for all $x\in I$, $\mu\in A$, we have $f(x)\geq\theta(x,\mu)$ for $x\in[0.5,1]$ and $\mu\in A$.

We now examine the case where $x\in[0,0.5)$ and $\mu\in A$. Clearly $\theta(0,\mu)=0$ and $f(0)=0$, so $f(x)\geq\theta(x,\mu)$ for $x=0$, $\mu\in A$. For $0<x<0.5$, define

$$\beta(x,\mu)=[f(x)-\theta(x,\mu)]/x$$

Then $\beta(x,\mu)=((16/3)-\mu)-(8-\mu)x+(16/3)x^3$ and $D_1\beta(x,\mu)=-(8-\mu)+16x^2<0$ for $0<x<0.5$, and $\beta(\cdot,\mu)$ is a decreasing function on $(0,0.5)$. As $x\uparrow(1/2)$, $\beta(x,\mu)\to 2-(\mu/2)\geq 0$. Thus, $\beta(x,\mu)>0$ for $0<x<(1/2)$. This means $f(x)>\theta(x,\mu)$ for $0<x<(1/2)$.

The welfare function

Recall that $u:I^2\times A\to\Re$ was defined by

$$u(x,z,\mu)=ax-0.5Lx^2+z\theta(x,\mu)-0.5z^2-\delta[az-0.5Lz^2+0.5\theta(z,\mu)^2]$$

We can compute the following derivatives:

$$D_1u(x,z,\mu)=a-Lx+z\mu(1-2x)$$
$$D_2u(x,z,\mu)=\mu x(1-x)-z-\delta a+\delta Lz-\delta\mu^2z+3\delta\mu^2z^2-2\delta\mu^2z^3$$
$$D_{11}u(x,z,\mu)=-L-2\mu z$$
$$D_{12}u(x,z,\mu)=\mu(1-2x)=D_{21}u(x,z,\mu)$$
$$D_{22}u(x,z,\mu)=-1+\delta L-\delta\mu^2+6\delta\mu^2z-6\delta\mu^2z^2$$

Recall that $\boldsymbol{D}\subset I^2$ was defined by

$$\boldsymbol{D}=\{(c,x)\in\Re_+\times I:c\leq f(x)\}$$

and $w: \boldsymbol{D} \times A \to \Re$ by

$$w(c, x, \mu) = u(x, f(x) - c, \mu) \quad \text{for } (c, x) \in \boldsymbol{D} \text{ and } \mu \in A$$

We check, first, that $w(\cdot, \mu)$ satisfies (W.1)–(W.4) on $\boldsymbol{D}$, given any $\mu \in A$. Clearly $w(\cdot, \mu)$ is C^2 on $\boldsymbol{D}$, and we check below that $w(\cdot, \mu)$ is strictly increasing in c (given x) and in x (given c), and strictly concave in (c, x) on $\boldsymbol{D}$. The first partials of w are:

$$D_1 w(c, x, \mu) = -D_2 u(x, f(x) - c, \mu)$$

$$D_2 w(c, x, \mu) = D_1 u(x, f(x) - c, \mu) + D_2 u(x, f(x) - c, \mu) f'(x)$$

Now, $D_2 u(x, f(x) - c, \mu) \leq (\mu/4) - \delta a + \delta z(L - \mu^2) - z + 3\delta\mu^2$

$$\leq (\mu/4) - \delta a + 3\delta\mu^2 \text{ [since } \delta(L - \mu^2) < 1]$$

$$\leq (\mu/4) - \delta[a - 3\mu^2]$$

$$< 0 \text{ [since } \delta[a - 3\mu^2] > (\mu/4)]$$

Thus, $D_1 w(c, x, \mu) > 0$ for $(c, x) \in \boldsymbol{D}$ and $\mu \in A$. Also, $D_1 u(x, z, \mu) \geq a - L - \mu \geq 323$ and

$$D_2 u(x, z, \mu) \geq -\delta a - z + \delta z(L - \mu^2) + \delta\mu^2 z^2(3 - 2z)$$

$$\geq -\delta a - z \text{ (since } L > \mu^2)$$

$$\geq -2.0625 > -3$$

and $f'(x) \leq f'(0) = (16/3)$. Thus, $D_2 u(x, z, \mu) f'(x) \geq -16$, and so $D_2 w(c, x, \mu) \geq 323. - 16 > 0$. Using the signs of the first partials of w, we conclude that $w(\cdot, \mu)$ is strictly increasing in c (given x) and in x (given c) on $\boldsymbol{D}$.

The second partials of w are:

$$D_{11} w(c, x, \mu) = D_{22} u(x, f(x) - c, \mu)$$

$$D_{12} w(c, x, \mu) = -D_{21} u(x, f(x) - c, \mu) - D_{22} u(x, f(x) - c, \mu) f'(x) = D_{21} w(c, x, \mu)$$

$$\begin{aligned} D_{22} w(c, x, \mu) = {} & D_{11} u(x, f(x) - c, \mu) + D_{12} u(x, f(x) - c, \mu) f'(x) \\ & + [D_{21} u(x, f(x) - c, \mu) + D_{22} u(x, f(x) - c, \mu) f'(x)] f'(x) \\ & + D_2 u(x, f(x) - c, \mu) f''(x) \end{aligned}$$

Now, $D_{22} u(x, z, \mu) \leq -1 + \delta L + 6\delta\mu^2 z \leq -1 + \delta(L + 96) < -0.5 < 0$, so we have

$$D_{11} w(c, x, \mu) < 0$$

Also,

$$\begin{aligned} (D_{11} w)(D_{22} w) - (D_{12} w)^2 = {} & (D_{22} u)(D_{11} u) + 2(D_{12} u)(D_{22} u) f'(x) \\ & + (D_{22} u)^2 f'(x)^2 + (D_{22} u)(D_2 u) f''(x) - (D_{12} u)^2 \\ & - (D_{22} u)^2 f'(x)^2 - 2(D_{12} u)(D_{22} u) f'(x) \\ = {} & (D_{22} u)(D_{11} u) - (D_{12} u)^2 + (D_{22} u)(D_2 u) f''(x) \end{aligned}$$

Now, $\quad D_{22} u(x, z, \mu) \geq -1 + \delta[L - \mu^2] > -1$

and $0 > D_2u(x,z,\mu) > -2.0625$, as we checked above. Further, $f''(x) = -16 + 64x^2$ for $0 \le x \le (1/2)$, so $0 \ge f''(x) \ge -16$ for $x \in I$. These estimates imply that

$$(D_2u)(D_{22}u)f''(x) > -33$$

We also have $D_{11}u(x,z,\mu) \le -L < 0$ and $D_{22}u(x,z,\mu) < -0.5 < 0$, while $[D_{12}u(x,z,\mu)]^2 = \mu^2(1-2x)^2 \le \mu^2 \le 16$. These estimates imply that

$$(D_{11}u)(D_{22}u) - (D_{12}u)^2 > (L/2) - 16 = 33$$

Thus, $(D_{11}w)(D_{22}w) - (D_{12}w)^2 > 33 - 33 = 0$ and $w(\cdot,\mu)$ is strictly concave on the set $\boldsymbol{D}$.

We have now checked that $w(\cdot,\mu)$ satisfies (W.1)–(W.4) on $\boldsymbol{D}$, given any $\mu \in A$. Furthermore, $w(\cdot,\mu)$ is C^2 on $\boldsymbol{D}$.

Recall that $Y \subset \mathfrak{R}^3_+$ was defined by

$$Y = \{(c,x) \in \mathfrak{R}^2_+ : c \le f(x)\}$$

and w was extended from $\boldsymbol{D}$ to Y by defining for $(c,x) \in Y$ with $x > 1$ [so that $f(x) = 1$, and $c \le 1$], and $\mu \in A$,

$$w(c,x,\mu) = w(c,1,\mu)$$

We now check that $w(\cdot,\mu)$ satisfies (W.1)–(W.4) on Y, given any $\mu \in A$. Note that $w(\cdot,\mu)$ satisfies (W.1) and (W.2) on Y. To check that $w(\cdot,\mu)$ satisfies (W.3), let (c,x) and $(\bar{c},\bar{x})$ belong to Y, and let $0 < \lambda < 1$. Then $w(\lambda(c,x) + (1-\lambda)(\bar{c},\bar{x}),\mu) = w(\lambda c + (1-\lambda)\bar{c}, \lambda x + (1-\lambda)\bar{x},\mu)$. Now, $[\lambda c + (1-\lambda)\bar{c},\ \lambda x + (1-\lambda)\bar{x}] \in \boldsymbol{D}$ if $[\lambda x + (1-\lambda)\bar{x}] \le 1$. If x and $\bar{x}$ are both ≤ 1, then concavity of w follows from concavity of w on $\boldsymbol{D}$. So consider, without loss of generality, that $x > 1$ while $\bar{x} < 1$. Then $w(\lambda x + (1-\lambda)\bar{x},\ \lambda c + (1-\lambda)\bar{c},\mu) \ge w(\lambda + (1-\lambda)\bar{x},\ \lambda c + (1-\lambda)\bar{c},\mu)$ [using (W.2)] $\ge \lambda w(1,c,\mu) + (1-\lambda)w(\bar{x},\bar{c},\mu)$ [since (1,c) and $(\bar{x},c)$ belong to D] $= \lambda w(x,c,\mu) + (1-\lambda)w(\bar{x},\bar{c},\mu)$. If $[\lambda x + (1-\lambda)\bar{x}] > 1$, then $w(\lambda c + (1-\lambda)\bar{c}, \lambda x + (1-\lambda)\bar{x},\mu) = w(\lambda c + (1-\lambda)\bar{c}, 1,\mu) \ge \lambda w(c,1,\mu) + (1-\lambda)w(\bar{c},1,\mu)$ [since $(c,1) \in \boldsymbol{D}$ and $(\bar{c},1) \in \boldsymbol{D}$] $\ge \lambda w(c,x,\mu) + (1-\lambda)w(\bar{c},\bar{x},\mu)$. Thus, $w(\cdot,\mu)$ satisfies (W.3) on Y.

To check (W.4), let $x \ge 0$ be given. Then if $x \le 1$, $w(c,x,\mu)$ is strictly increasing and strictly concave in c. If $x > 1$, $w(c,x,\mu) = w(c,1,\mu)$, which is strictly concave and strictly increasing in c.

Recall that $w(\cdot,\mu)$ was extended from Y to $\mathfrak{R}^2_+$ by defining for $(c,x) \in \mathfrak{R}^2_+$ with $c > f(x)$, and $\mu \in A$,

$$w(c,x,\mu) = w(f(x),x,\mu)$$

We now check that $w(\cdot,\mu)$ satisfies (W.1)–(W.4), given any $\mu \in A$. Note that $w(\cdot,\mu)$ clearly satisfies (W.1), (W.2) on $\mathfrak{R}^2_+$. To check (W.3), let (c,x) and $(\bar{c},\bar{x})$ be in $\mathfrak{R}^2_+$ and let $0 < \lambda < 1$. Then,

$$w(\lambda(c,x) + (1-\lambda)(\bar{c},x),\mu) = w(\lambda c + (1-\lambda)\bar{c}, \lambda x + (1-\lambda)\bar{x},\mu)$$

Denote $\min[c,f(x)]$ by $G(c,x)$; $\min[\bar{c},f(\bar{x})]$ by $G(\bar{c},\bar{x})$. Then, we have $w(\lambda c + (1-\lambda)\bar{c}, \lambda x + (1-\lambda)\bar{x},\mu) \ge w(\lambda G(c,x) + (1-\lambda)G(\bar{c},\bar{x}),\ \lambda x + (1-\lambda)\bar{x},\mu)$. Also $\lambda G(c,x) + (1-\lambda)G(\bar{c},\bar{x}) \le \lambda f(x) + (1-\lambda)f(\bar{x}) \le f(\lambda x + (1-\lambda)\bar{x})$. Further, $G(c,x) \le f(x)$ and $G(\bar{c},\bar{x}) \le f(\bar{x})$. Thus, $(G(c,x),x)$, $(G(\bar{c},\bar{x}),\bar{x})$, and $[\lambda G(c,x) + (1-\lambda)G(\bar{c},\bar{x}),\ \lambda x +$

$(1-\lambda)\bar{x}]$ all belong to Y and so we have

$$w(\lambda c+(1-\lambda)\bar{c}, \lambda x+(1-\lambda)\bar{x}, \mu) \geq \lambda w(G(c,x),x,\mu)+(1-\lambda)w(G(\bar{c},\bar{x}),\bar{x},\mu).$$

If $\min(c, f(x)) = c$, then $w(G(c,x),x,\mu) = w(c,x,\mu)$; if $\min(c, f(x)) \neq c$, then $c > f(x)$, and $w(G(c,x),x,\mu) = w(f(x),x,\mu) = w(c,x,\mu)$; thus, in either case, $w(G(c,x),x,\mu) = w(c,x,\mu)$. Similarly, $w(G(\bar{c},\bar{x}),\bar{x},\mu) = w(\bar{c},\bar{x},\mu)$. Hence, $\lambda w(G(c,x),x,\mu)+(1-\lambda)\times w(G(\bar{c},\bar{x},\mu)) = \lambda w(c,x,\mu)+(1-\lambda)w(\bar{c},\bar{x},\mu)$, completing our demonstration of the concavity of $w(\cdot,\mu)$ on $\Re^2_+$.

Appendix B
Verification of optimal transition function

The verification of the optimal transition function is obtained by following closely the technique of Boldrin–Montruchhio (1986).

Define $\phi: I^2 \times A \to \Re$ by

$$\phi(x,z,\mu) = ax - 0.5Lx^2 + z\mu x(1-x) - 0.5z^2$$

We can compute the following derivatives:

$$\begin{aligned} D_1\phi(x,z,\mu) &= a - Lx + z\mu(1-2x) \\ D_2\phi(x,z,\mu) &= \mu x(1-x) - z \\ D_{11}\phi(x,z,\mu) &= -L - 2\mu z \\ D_{12}\phi(x,z,\mu) &= \mu(1-2x) = D_{21}\phi(x,z,\mu) \\ D_{22}\phi(x,z)| &= -1 \end{aligned}$$

Looking at the Hessian matrix of $\phi(\cdot,\mu)$, we note that

$$D_{11}\phi(x,z,\mu) < 0 \quad [\text{since } L > 0]$$

and

$$(D_{11}\phi)(D_{22}\phi) - (D_{12}\phi)^2 = L + 2\mu z - [\mu(1-2x)]^2 > 0$$
$$[\text{since } |1-2x| \leq 1 \text{ and } |\mu| \leq 4, \text{ and } L > 16]$$

Thus $\phi(\cdot,\mu)$ is strictly concave on I^2.

Result 1

Given any $x \in I$, $\theta(x,\mu) \equiv \mu x(1-x)$ solves the problem:

$$\left.\begin{array}{l} \text{Max } \phi(x,z,\mu) \\ \text{Subject to } z \in I \end{array}\right\} \quad (P)$$

Furthermore, this solution is unique.

Proof

Clearly, $\theta(x,\mu) \in I$ for all $x \in I$ and $\mu \in A$. For any $z \in I$ with $z \neq \theta(x,\mu)$ we have by the strict concavity of $\phi(\cdot,\mu)$ on I^2,

$$\phi(x,z,\mu) - \phi(x,\theta(x,\mu),\mu) < D_2\phi(x,\theta(x,\mu),\mu)(z-\theta(x,\mu)) = 0$$

Thus, $\theta(x,\mu)$ uniquely solves problem (P).

Define $\psi: I \times A \to I$ by

$$\psi(x, \mu) = \phi(x, \theta(x, \mu), \mu) \quad \text{for } (x, \mu) \in I \times A$$

Then, by the definition of ϕ and θ,

$$\psi(x, \mu) = ax - 0.5Lx^2 + \theta(x, \mu)^2 - 0.5\theta(x, \mu)^2 = ax - 0.5Lx^2 + 0.5\theta(x, \mu)^2$$

Note that by the definition of $u(x, z, \mu)$,

$$\begin{aligned} u(x, z, \mu) &= ax - 0.5Lx^2 + z\theta(x, \mu) - 0.5z^2 - \delta[az - 0.5Lx^2 + 0.5\theta(z, \mu)^2] \\ &= \phi(x, z, \mu) - \delta\psi(z, \mu) \end{aligned}$$

for $(x, z) \in I^2$ and $\mu \in A$.

Result 2

Given any $x \in I$, (i) $\theta(x, \mu)$ solves the problem

$$\left.\begin{aligned} &\text{Max } u(x, z, \mu) + \delta\psi(z, \mu) \\ &\text{Subject to } z \in I \end{aligned}\right\} \quad (Q)$$

and (ii) $\psi(x, \mu) = \underset{z \in I}{\text{Max}} [u(x, z, \mu) + \delta\psi(z, \mu)]$

Proof

We have $u(x, z, \mu) + \delta\psi(z, \mu) = \phi(x, z, \mu)$ by definition of u. Thus (i) follows from Result 1.

Using (i), we get

$$\begin{aligned} \underset{z \in I}{\text{Max}} [u(x, z, \mu) + \delta\psi(z, \mu)] &= u(x, \theta(x, \mu), \mu) + \delta\psi(\theta(x, \mu), \mu) \\ &= \phi(x, \theta(x, \mu), \mu) \quad \text{[by definition of } u\text{]} \\ &= \psi(x, \mu) \quad \text{[by definition of } \psi\text{]} \end{aligned}$$

This proves (ii). //

Consider a sequence $(\hat{x}_t)_0^\infty$ defined by $\hat{x}_0 = x \in I$, $\hat{x}_{t+1} = \theta(\hat{x}_t, \mu)$ for $t \geq 0$. Notice that $\hat{x}_t \in I$ for $t \geq 0$. Also, for $t \geq 0$,

$$\begin{aligned} \psi(\hat{x}_t, \mu) &= \phi(\hat{x}_t, \theta(\hat{x}_t, \mu), \mu) \\ &= \phi(\hat{x}_t, \hat{x}_{t+1}, \mu) \\ &= u(\hat{x}_t, \hat{x}_{t+1}, \mu) + \delta\psi(\hat{x}_{t+1}, \mu) \end{aligned}$$

Thus, iterating on this relation,

$$\psi(x, \mu) = \sum_{t=0}^{T} \delta^t u(\hat{x}_t, \hat{x}_{t+1}, \mu) + \delta^T \psi(\hat{x}_{T+1}, \mu)$$

Since $\psi(\cdot, \mu)$ is bounded on I and $u(\cdot, \mu)$ is bounded on I^2, and $0 < \delta < 1$, we get

$$\psi(x, \mu) = \sum_{t=0}^{\infty} \delta^t u(\hat{x}_t, \hat{x}_{t+1}, \mu) \tag{1}$$

Consider, next, any sequence $(x_t)_0^\infty$ defined by $x_0 = x \in I$, $(x_t, x_{t+1}) \in I^2$ for $t \geq 0$.

Then, by using Result 2,

$$\psi(x_t, \mu) \geq u(x_t, x_{t+1}, \mu) + \delta\psi(x_{t+1}, \mu)$$

Iterating on this relation

$$\psi(x, \mu) \geq \sum_{t=0}^{T} \delta^t u(x_t, x_{t+1}, \mu) + \delta^T \psi(x_{T+1}, \mu)$$

Again, using the boundedness of $u(\cdot, \mu)$ on I^2 and of $\psi(\cdot, \mu)$ on I, and $0 < \delta < 1$,

$$\psi(x, \mu) \geq \sum_{t=0}^{\infty} \delta^t u(x_t, x_{t+1}, \mu) \tag{2}$$

Given (1) and (2), and the strict concavity of $u(\cdot, \mu)$ on I^2, we can conclude

$$\sum_{t=0}^{\infty} \delta^t u(\hat{x}_t, \hat{x}_{t+1}, \mu) = \psi(x, \mu) > \sum_{t=0}^{\infty} \delta^t u(x_t, x_{t+1}, \mu) \tag{3}$$

for every $(x_t)_0^\infty$ satisfying $x_0 = x$, $(x_t, x_{t+1}) \in I^2$ for $t \geq 0$, and $x_t \neq \hat{x}_t$ for some $t \geq 1$.

Note that $(\hat{x}_t)$ is a *program* from $x \in I$, since $\hat{x}_0 = x$, $\hat{x}_t \geq 0$ for $t \geq 1$ and $\hat{x}_t = \theta(\hat{x}_{t-1}, \mu) \leq f(\hat{x}_{t-1})$ for $t \geq 1$. Also, by definition of w, we have

$$\sum_0^\infty \delta^t w(f(\hat{x}_t) - \hat{x}_{t+1}, \hat{x}_t, \mu) = \sum_0^\infty \delta^t u(\hat{x}_t, \hat{x}_{t+1}, \mu) \tag{4}$$

If (x_t) is any program from $x \in I$, then $x_0 = x$, $x_t \in I$ for $t \geq 0$, and

$$\sum_0^\infty \delta^t w(f(x_t) - x_{t+1}, x_t, \mu) = \sum_0^\infty \delta^t u(x_t, x_{t+1}, \mu) \tag{5}$$

Using (3), (4) and (5), we conclude that

$$\sum_0^\infty \delta^t w(\hat{c}_{t+1}, \hat{x}_t, \mu) > \sum_0^\infty \delta^t w(c_{t+1}, x_t, \mu) \tag{6}$$

for every program (x_t) from x, for which $x_t \neq \hat{x}_t$ for some $t \geq 1$. Thus $(\hat{x}_t)$ is the unique μ-optimal program from x. Consequently, the optimal transition function $h_\mu(x)$ satisfies

$$h_\mu(x) = \mu x(1 - x) \quad \text{for all } x \in I,\ \mu \in A$$

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